CLASSIFICATION OF SEMISTABLE SHEAVES ON A RATIONAL CURVE WITH ONE NODE

SERGEY MOZGOVOY

Abstract. We classify (semi)stable sheaves on a rational curve with one node. The results are based on the classification of indecomposable torsion-free sheaves due to Drozd and Greuel [4], where the sheaves are described in terms of certain combinatorial data. We translate the condition of (semi)stability into this combinatorial language and solve the so obtained problem.

1. Introduction

A rational curve with one node is among those few examples of singular curves, where a complete classification of indecomposable vector bundles is possible. The classification was done by Drozd and Greuel in [4] in terms of certain combinatorial objects, which are easy to handle. Using their technique one can also classify all indecomposable torsion-free sheaves.

A next natural question would be to describe all semistable and stable sheaves. However, until now only some partial results in this direction were known. The problem was solved for the sheaves of degree 0 by Burban and Kreußler [3, 2] by reducing it to the classification of torsion sheaves made by Gelfand and Ponomarev [6]. In the case of coprime degree and rank Burban [1] classified stable locally free sheaves by detecting those sheaves that have a one-dimensional endomorphism ring.

In this paper we give a classification of indecomposable (semi)stable sheaves (of nonzero rank) on a rational curve with one node over an algebraically closed field of characteristic 0. In order to do this we introduce and analyze certain combinatorial objects — chains and cycles (see Definition 2.1), which are used for the classification of indecomposable torsion-free sheaves. With any aperiodic cycle $\mathbf{a}$ one associates an indecomposable locally free sheaf $\mathcal{B}(\mathbf{a})$ (see [4, 3] and Section 2) and with any chain $\mathbf{b}$ one associates an indecomposable non-locally free sheaf $\mathcal{S}(\mathbf{b})$ (see [3] and Section 2). The conditions of (semi)stability of $\mathcal{B}(\mathbf{a})$ and $\mathcal{S}(\mathbf{b})$ imply certain conditions on the cycle $\mathbf{a}$ and the chain $\mathbf{b}$. We will call these conditions the conditions of (semi)stability of cycles and chains, respectively (see Section 3 for precise definitions). One of the main results of the paper is

Theorem 1.1. *Given an aperiodic cycle $\mathbf{a}$, the sheaf $\mathcal{B}(\mathbf{a})$ is (semi)stable if and only if the cycle $\mathbf{a}$ is (semi)stable. Given a chain $\mathbf{b} = (b_1, \dots, b_r)$, the sheaf $\mathcal{S}(\mathbf{b})$ is (semi)stable if and only if the chain $(b_1 + 1, b_2, \dots, b_{r-1}, b_r + 1)$ is (semi)stable.*

This means that we only need to classify the (semi)stable chains and cycles. This is a purely combinatorial problem, and it has the following solution. Let $M^{ss}_{\mathrm{cyc}}(r, d)$ (respectively, $M^{s}_{\mathrm{cyc}}(r, d)$, $M^{ss}_{\mathrm{ch}}(r, d)$, $M^{s}_{\mathrm{ch}}(r, d)$) be the set of all aperiodic semistable

2000 *Mathematics Subject Classification.* 14H60,14D20.

cycles (respectively, aperiodic stable cycles, semistable chains, stable chains) of rank r and degree d.

Theorem 1.2. *Let $r \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}$. Then*

(1) *There is a natural bijection between $M^{ss}_{\mathrm{ch}}(r,d)$ and $M^{ss}_{\mathrm{ch}}(r,d+r)$ and if $0 < d < r$ then there is a natural bijection between $M^{ss}_{\mathrm{ch}}(r,d)$ and $M^{ss}_{\mathrm{ch}}(d,d-r)$. As a corollary, there is a bijection between $M^{ss}_{\mathrm{ch}}(r,d)$ and $M^{ss}_{\mathrm{ch}}(h,0)$, where $h = \gcd(r,d)$. The same assertions hold for stable chains, stable cycles and semistable cycles.*

(2) *The set $M^{ss}_{\mathrm{ch}}(r,d)$ is finite and non-empty. If r and d are coprime then $M^{s}_{\mathrm{ch}}(r,d) = M^{ss}_{\mathrm{ch}}(r,d)$ and it contains just one element. Otherwise, $M^{s}_{\mathrm{ch}}(r,d)$ is empty. The same assertions hold for cycles.*

Thus the description of $M^{ss}_{\mathrm{ch}}(r,d)$ is reduced to the description of $M^{ss}_{\mathrm{ch}}(h,0)$, $h = \gcd(r,d)$, and the latter is given in Proposition 3.11 (for the analogous classification of cycles, see Proposition 3.18). Among other things, we prove that for $h > 1$ there are no stable chains (cycles) and for $h = 1$ there is just one semistable chain (cycle) which is actually stable. This implies the second part of Theorem 1.2.

In Section 2 we recall the definition of indecomposable sheaves associated to chains and cycles. Writing down the (semi)stability condition for these sheaves we get certain conditions on chains and cycles, which we call the (semi)stability conditions on chains and cycles.

In Section 3 we describe basic properties of chains and cycles. We define (semi) stable chains and cycles, analyze their structure and give basic reduction methods. Altogether this allows us to classify the (semi)stable chains and cycles.

In Section 4 we use the classification from Section 3 to prove that for any (semi)stable chain or cycle, the associated sheaf is also (semi)stable. Together with the results from Section 2 this proves that the conditions of (semi)stability of chains and cycles are necessary and sufficient for the (semi)stability of the corresponding sheaves. As a corollary, we prove in particular that any indecomposable semistable sheaf is homogeneous, i.e., all stable factors of its Jordan-Hölder filtration are isomorphic. This was proved in [5] for sheaves of degree 0.

My cordial thanks go to N.Sidorova. Proposition 3.10 is a result of our communication. I would like to thank also I.Burban, M.Lehn, C. Pillau, C. Sorger, I.Yudin for many useful comments.

2. Semistable sheaves

Let C be a rational curve with one node over an algebraically closed field k of characteristic 0. Let $\pi : \widetilde{C} \longrightarrow C$ be its normalization ($\widetilde{C} \simeq \mathbb{P}^1$). Given a torsion-free sheaf F over C of rank r and degree d ($\deg F = \chi(F) - r\chi(\mathcal{O}_C) = \chi(F)$), we will say that F is of type (r,d). We will denote the set of torsion-free, indecomposable sheaves of type (r,d) by $\mathcal{E}(r,d)$. Such sheaves were classified by Drozd and Greuel [4] in terms of certain combinatorial data. Our aim is to express the conditions of stability and semistability of sheaves in $\mathcal{E}(r,d)$ in the language of this combinatorial data. As a result we will get a classification of stable and semistable sheaves.

Let us give a description of sheaves in $\mathcal{E}(r,d)$ according to [4]. Let p, p^* be preimages of a singular point in C under π. For any line bundle $L \simeq \mathcal{O}(n)$ over $\widetilde{C}$ we fix once and for all the bases of the fibers $L(p)$ and $L(p^*)$. To make possibly few choices we do this in the following way. Fix some section s of $\mathcal{O}(1)$ having zero in

some point different from p and p^*. Then s^n will induce nonzero elements of the fibers of $\mathcal{O}(n)$ over p and p^*, giving the necessary bases.

Definition 2.1. Define a chain to be a finite sequence of integers. Define a cycle to be an equivalence class of chains, where the equivalence is generated by relations

$$(a_1, a_2, \dots, a_r) \sim (a_2, a_3, \dots, a_r, a_1).$$

We will usually write representing sequences instead of the corresponding cycles.

Given a finite sequence of integers $\mathbf{a} = (a_1, \dots, a_r)$, a natural number m and an element $\lambda \in k^*$, we construct the vector bundle $\mathcal{B}(\mathbf{a}, m, \lambda)$ over C in the following way (see [4] or [3] for more formal description). Consider the sheaves $B_i = \mathcal{O}(a_i)^{\oplus m}$ over $\widetilde{C}$, then take the direct image π_* of their sum and make the following identifications over the singular point: glue $B_1(p^*)$ with $B_2(p)$, glue $B_2(p^*)$ with $B_3(p)$ and so on up to identification of $B_r(p^*)$ with $B_1(p)$. The gluing matrices (with respect to the above chosen bases) are defined to be unit matrices except the matrix gluing $B_r(p^*)$ with $B_1(p)$ which is defined to be a Jordan block of size m with an eigenvalue λ.

Remark 2.2. Note that if $\mathbf{a} = (a_1, \dots, a_r)$ and $\mathbf{a}' = (a_2, \dots, a_r, a_1)$ is its cyclic shift, then $\mathcal{B}(\mathbf{a}, m, \lambda) \simeq \mathcal{B}(\mathbf{a}', m, \lambda)$. This means that the vector bundle $\mathcal{B}(\mathbf{a}, m, \lambda)$ is determined by the cycle $\mathbf{a}$ (together with $m \in \mathbb{Z}_{>0}$, $\lambda \in k^*$).

Theorem 2.3 (see [4])**.** *The sheaves $\mathcal{B}(\mathbf{a}, m, \lambda)$ with aperiodic (see Definition 3.3) cycles $\mathbf{a}$ describe all indecomposable locally free sheaves over C. Different cycles induce non-isomorphic sheaves.*

In particular, consider the cycle $\mathbf{0} = (0)$ and define $F_m := \mathcal{B}(\mathbf{0}, m, 1)$. One can show that $F_1 \simeq \mathcal{O}_C$ and there is an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow F_m \longrightarrow F_{m-1} \longrightarrow 0.$$

Lemma 2.4 (see [7])**.** *There are isomorphisms*

$$\mathcal{B}(\mathbf{a}, m, \lambda) \simeq \mathcal{B}(\mathbf{a}, 1, \lambda) \otimes F_m, \qquad \mathcal{B}(\mathbf{a}, 1, \lambda^r) \simeq \mathcal{B}(\mathbf{a}, 1, 1) \otimes \mathcal{B}(\mathbf{0}, 1, \lambda),$$

where r is the length of $\mathbf{a}$.

We will denote $\mathcal{B}(\mathbf{a}, 1, 1)$ by $\mathcal{B}(\mathbf{a})$.

Corollary 2.5. *The sheaf $\mathcal{B}(\mathbf{a}, m, \lambda)$ is semistable if and only if $\mathcal{B}(\mathbf{a})$ is semistable. The sheaf $\mathcal{B}(\mathbf{a}, m, \lambda)$ is stable if and only if $m = 1$ and $\mathcal{B}(\mathbf{a})$ is stable.*

Proof. We know that the sheaf F_m has a filtration with factors isomorphic to $\mathcal{O}_C$. Therefore the sheaf $\mathcal{B}(\mathbf{a}, m, \lambda)$ has a filtration with factors isomorphic to $\mathcal{B}(\mathbf{a}, 1, \lambda)$. Hence $\mathcal{B}(\mathbf{a}, m, \lambda)$ is semistable if and only if $\mathcal{B}(\mathbf{a}, 1, \lambda)$ is semistable and $\mathcal{B}(\mathbf{a}, m, \lambda)$ can be stable just if $m = 1$. It is clear that $\mathcal{B}(\mathbf{a}, 1, \lambda)$ is (semi)stable if and only if $\mathcal{B}(\mathbf{a}, 1, 1)$ is. $\square$

Let us now describe the non-locally free indecomposable sheaves. Given a chain $\mathbf{a} = (a_1, \dots, a_r)$, we define the torsion-free sheaf $\mathcal{S}(\mathbf{a})$ as follows. Take the direct image π_* of the sum of $B_i = \mathcal{O}(a_i)$ and make the following identifications over the singular point: glue $B_1(p^*)$ with $B_2(p)$, glue $B_2(p^*)$ with $B_3(p)$ and so on, identifying their bases. The fibers $B_1(p)$ and $B_r(p^*)$ are not identified.

Theorem 2.6. *The sheaves $\mathcal{S}(\mathbf{a})$ describe all indecomposable torsion-free non-locally free sheaves over C. Different chains induce non-isomorphic sheaves.*

Our goal is to determine which of the sheaves $\mathcal{B}(\mathbf{a}, m, \lambda)$ and $\mathcal{S}(\mathbf{a})$ are (semi)stable. It follows from Corollary 2.5 that in the case of locally free sheaves we can restrict ourselves just to $\mathcal{B}(\mathbf{a})$.

Remark 2.7. Given a chain $\mathbf{a} = (a_1, \dots, a_r)$, one can easily show that $\deg \mathcal{B}(\mathbf{a}) = \sum a_i$, $\deg \mathcal{S}(\mathbf{a}) = \sum a_i + 1$ and $\operatorname{rk} \mathcal{B}(\mathbf{a}) = \operatorname{rk} \mathcal{S}(\mathbf{a}) = r$.

Proposition 2.8. *Let $\mathbf{a} = (a_1, \dots, a_r)$ be a cycle and $\mathbf{b} = (b_1, \dots, b_k)$ be its subchain (see Definition 3.1). Then there is an exact sequence*

$$0 \longrightarrow \mathcal{S}((b_1 - 1, b_2, \dots, b_{k-1}, b_k - 1)) \longrightarrow \mathcal{B}(\mathbf{a}) \longrightarrow \mathcal{S}(\mathbf{b}') \longrightarrow 0,$$

where for $k = 1$ we consider just $\mathcal{S}((b_1 - 2))$ and $\mathbf{b}'$ is the complement of $\mathbf{b}$ in $\mathbf{a}$.

Proof. Without loss of generality we may assume $\mathbf{b} = (a_1, \dots, a_k)$. Let us denote $B_i = \mathcal{O}_{\widetilde{C}}(a_i)$. We consider the direct image π_* of the sum

$$(B_1 \otimes \mathcal{O}_{\widetilde{C}}(-p)) \oplus B_2 \oplus \dots \oplus B_{k-1} \oplus (B_k \otimes \mathcal{O}_{\widetilde{C}}(-p^*))$$

and identify their fibers precisely like in the construction of $\mathcal{S}$. The module obtained in this way is isomorphic to $\mathcal{S}((b_1 - 1, b_2, \dots, b_{k-1}, b_k - 1))$ and there is a natural embedding of this module to $\mathcal{B}(\mathbf{a})$ (the fiber of $(B_1 \otimes \mathcal{O}_{\widetilde{C}}(-p))$ in point p goes to zero both in fibers $B_r(p^*)$ and $B_1(p)$, and analogously for the fiber of $(B_k \otimes \mathcal{O}_{\widetilde{C}}(-p^*))$ in point p^*). It is clear that the quotient is isomorphic to the direct image of $B_{k+1} \oplus \dots \oplus B_r$ with identifications $B_{k+1}(p^*) \simeq B_{k+2}(p), \dots B_{r-1}(p^*) \simeq B_r(p)$. But such a module is precisely $\mathcal{S}((a_{k+1}, \dots, a_r))$. □

Corollary 2.9. *Let $\mathbf{a} = (a_1, \dots, a_r)$ be a cycle such that $\mathcal{B}(\mathbf{a})$ is a semistable sheaf. Then for any proper subchain $\mathbf{b} = (b_1, \dots, b_k)$ of $\mathbf{a}$ it holds*

$$\frac{\sum_{i=1}^k b_i - 1}{k} \leq \frac{\sum_{i=1}^r a_i}{r}.$$

If $\mathcal{B}(\mathbf{a})$ is stable then the inequalities are strict.

Proposition 2.10. *Let $\mathbf{a} = (a_1, \dots, a_r)$ be a chain and $\mathbf{b} = (b_1, \dots, b_k)$ be its subchain that does not contain a_1 and a_r. Then there is an embedding*

$$\mathcal{S}((b_1 - 1, b_2, \dots, b_{k-1}, b_k - 1)) \hookrightarrow \mathcal{S}(\mathbf{a}).$$

If $\mathbf{b}$ is a subchain containing a_1 or a_r (say, $\mathbf{b} = (a_1, \dots, a_k)$) then there is an exact sequence

$$0 \longrightarrow \mathcal{S}((a_1, a_2, \dots, a_{k-1}, a_k - 1)) \longrightarrow \mathcal{S}(\mathbf{a}) \longrightarrow \mathcal{S}((a_{k+1}, \dots, a_r)) \longrightarrow 0.$$

Proof. The proof goes through the same lines as the proof of Proposition 2.8. □

Corollary 2.11. *Let $\mathbf{a} = (a_1, \dots, a_r)$ be a chain such that $\mathcal{S}(\mathbf{a})$ is a semistable sheaf and let*

$$\mathbf{a}' := (a_1 + 1, a_2, \dots, a_{r-1}, a_r + 1).$$

Then for any proper subchain $\mathbf{b} = (b_1, \dots, b_k)$ of $\mathbf{a}'$ it holds

$$\frac{\sum_{i=1}^k b_i - 1}{k} \leq \frac{\sum_{i=1}^r a'_i - 1}{r}.$$

If $\mathcal{S}(\mathbf{a})$ is stable then the inequalities are strict.

Proof. If $\mathcal{S}(\mathbf{a})$ is semistable then for any subchain $\mathbf{b} = (b_1, \dots, b_k)$ of $\mathbf{a}'$ that does not contain a'_1 and a'_r we have (see Proposition 2.10 and Remark 2.7)

$$\frac{\sum_{i=1}^k b_i - 2 + 1}{k} \le \frac{\sum_{i=1}^r a_i + 1}{r} = \frac{\sum_{i=1}^r a'_i - 1}{r}.$$

If $\mathbf{b}$ is a subchain of $\mathbf{a}'$ containing a'_1 or a'_r (say, $\mathbf{b} = (a'_1, \dots, a'_k)$) then according to Proposition 2.10 there is an embedding $\mathcal{S}((b_1 - 1, b_2, \dots, b_{k-1}, b_k - 1)) \hookrightarrow \mathcal{S}(\mathbf{a})$ and this implies

$$\frac{\sum_{i=1}^k b_i - 2 + 1}{k} \le \frac{\sum_{i=1}^r a_i + 1}{r} = \frac{\sum_{i=1}^r a'_i - 1}{r}.$$

The claim about stability is analogous. □

Corollaries 2.9 and 2.11 suggest that one can define stability conditions directly for chains and cycles. We will do this in the next section. After the classification of (semi)stable chains and cycles we will be able to prove that the stability of chains and cycles is not only necessary for the stability of the corresponding sheaves (as it is proved in Corollaries 2.9 and 2.11), but is also sufficient.

3. Semistable chains and cycles

Recall from Definition 2.1 that chains are finite sequences of integers and cycles are equivalence classes of chains with respect to the cyclic shift.

Definition 3.1. Given a chain $(a_1, \dots, a_r)$, define its subchain as any chain of the form $(a_i, a_{i+1}, \dots, a_j)$, where $1 \le i \le j \le r$. Given a cycle $(a_1, \dots, a_r)$, define its subchain as any chain of the form $(a_i, a_{i+1}, \dots, a_{i+k})$, where $1 \le i \le r$, $0 \le k < r$ and we identify a_{r+1} with a_1, a_{r+2} with a_2 and so on.

For example, the cycle $(1, 2, 3, 1, 2, 3)$ contains the subchain $(3, 1, 2, 3, 1)$ but the chain $(1, 2, 3, 1, 2, 3)$ does not.

Definition 3.2. Given a chain $\mathbf{a} = (a_1, \dots, a_r)$, we call any of its subchains containing a_1 or a_r an extreme subchain of $\mathbf{a}$.

Definition 3.3. A cycle is called aperiodic if its sequence cannot be written as a concatenation of equal proper subsequences.

Definition 3.4. Given a chain $\mathbf{a} = (a_1, \dots, a_r)$, define its degree, rank, and slope by

$$\deg \mathbf{a} = \sum_{i=1}^r a_i - 1, \qquad \operatorname{rk} \mathbf{a} = r, \qquad \mu(\mathbf{a}) = \frac{\deg \mathbf{a}}{\operatorname{rk} \mathbf{a}}.$$

Given a cycle $\mathbf{a} = (a_1, \dots, a_r)$, define its degree, rank, and slope by

$$\deg \mathbf{a} = \sum_{i=1}^r a_i, \qquad \operatorname{rk} \mathbf{a} = r, \qquad \mu(\mathbf{a}) = \frac{\deg \mathbf{a}}{\operatorname{rk} \mathbf{a}}.$$

For example, the slope of the chain $(1, 2, 3, 1, 2, 3)$ equals $\frac{11}{6}$ and the slope of the cycle $(1, 2, 3, 1, 2, 3)$ equals 2.

Definition 3.5. The chain (cycle) $\mathbf{a} = (a_1, \dots, a_r)$ is called semistable if for any its subchain $\mathbf{b}$ it holds

$$\mu(\mathbf{b}) \le \mu(\mathbf{a}).$$

If the inequality is strict for any proper subchain then $\mathbf{a}$ is called stable. A proper subchain $\mathbf{b}$ of a chain (cycle) $\mathbf{a}$ is called a destabilizing subchain of $\mathbf{a}$ if $\mu(\mathbf{b}) \geq \mu(\mathbf{a})$.

For example, the chain $(1,0,0,1)$ is stable and the cycle $(1,0,0,1)$ is not stable, because it has slope $1/2$ and contains a destabilizing subchain $(1,1)$ having the same slope. In what follows, we will classify (semi)stable chains and cycles. For example, the only stable chain of rank 7 and degree 4 is $(1,1,0,1,0,1,1)$ and the only stable (aperiodic) cycle of rank 7 and degree 4 is $(1,0,1,0,1,0,1)$.

A chain (cycle) of rank r and degree d will be said to be of type (r,d). We want to classify all (semi)stable chains and aperiodic cycles of a fixed type (r,d). The set of semistable (respectively, stable) chains of type (r,d) will be denoted by $M^{ss}_{\mathrm{ch}}(r,d)$ (respectively, $M^{s}_{\mathrm{ch}}(r,d)$). The set of aperiodic semistable (stable) cycles of type (r,d) will be denoted by $M^{ss}_{\mathrm{cyc}}(r,d)$ ($M^{s}_{\mathrm{cyc}}(r,d)$). The study of semistable chains and semistable cycles is quite analogous but we will deal with them separately.

Lemma 3.6. *A chain $\mathbf{a} = (a_1, a_2, \dots a_r)$ is (semi)stable if and only if the chain $(a_1+1, a_2+1 \dots, a_r+1)$ is (semi)stable. In particular, there is a natural bijection $M^{ss}_{\mathrm{ch}}(r,d) \simeq M^{ss}_{\mathrm{ch}}(r,d+r)$ and we can always assume $0 \leq \deg \mathbf{a} < r$.*

Lemma 3.7. *For any subchain $\mathbf{b}$ of a semistable chain $\mathbf{a} = (a_1,\dots,a_r)$ one has*

$$\mu(\mathbf{a}) \leq \frac{\sum b_i + 1}{\operatorname{rk} \mathbf{b}}.$$

Moreover, if $\mathbf{b}$ is an extreme subchain then

$$\mu(\mathbf{a}) \leq \frac{\sum b_i}{\operatorname{rk} \mathbf{b}}.$$

Proof. Let us first prove the assertion for extreme subchains. We may assume that $\mathbf{b} = (a_1,\dots,a_{k_1})$. Denote $k_2 = r - k_1$, $x_1 = \sum_{i=1}^{k_1} a_i$, $x_2 = \sum_{i=k_1+1}^{r} a_i$. Then the semistability of $\mathbf{a}$ implies

$$\frac{x_2 - 1}{k_2} \leq \frac{x_1 + x_2 - 1}{k_1 + k_2},$$

hence

$$\frac{x_1 + x_2 - 1}{k_1 + k_2} \leq \frac{x_1}{k_1}.$$

Let us now assume that $\mathbf{b} = (a_{k_1+1},\dots,a_{k_1+k_2})$ is not extreme, i.e., $k_1 \geq 1$ and $k_3 := r - k_1 - k_2 \geq 1$. We denote $x_1 = \sum_{i=1}^{k_1} a_i$, $x_2 = \sum_{i=k_1+1}^{k_1+k_2} a_i$, and $x_3 = \sum_{i=k_1+k_2+1}^{r} a_i$. It holds by our assumptions

$$\frac{x_1 - 1}{k_1} \leq \frac{x_1 + x_2 + x_3 - 1}{k_1 + k_2 + k_3}, \qquad \frac{x_3 - 1}{k_3} \leq \frac{x_1 + x_2 + x_3 - 1}{k_1 + k_2 + k_3},$$

hence

$$\frac{x_1 + x_3 - 2}{k_1 + k_3} \leq \frac{x_1 + x_2 + x_3 - 1}{k_1 + k_2 + k_3}$$

and therefore

$$\frac{x_1 + x_2 + x_3 - 1}{k_1 + k_2 + k_3} \leq \frac{x_2 + 1}{k_2}.$$

□

Corollary 3.8. *If a chain $\mathbf{a} = (a_1,\dots,a_r)$ is semistable then $\mu(\mathbf{a}) \leq a_1$, $\mu(\mathbf{a}) \leq a_r$ and for any k one has $\mu(\mathbf{a}) \leq a_k + 1$.*

It follows that in a semistable chain $(a_1, \dots, a_r)$ for any indices i, j one has $a_i - 1 \le \mu(a) \le a_j + 1$ and therefore the difference between any a_i and a_j is not greater than 2. Hence, the elements of $\mathbf{a}$ can take at most 3 consecutive values.

Lemma 3.9. *If a semistable chain $\mathbf{a} = (a_1, \dots, a_r)$ of type (r, d) contains elements with difference 2 then d is a multiple of r (hence $d = 0$ under the assumption $0 \le d < r$).*

Proof. Assume there are elements in $\mathbf{a}$ equal to $m - 1$ and $m + 1$. Then we have $(m + 1) - 1 \le \mu(\mathbf{a}) \le (m - 1) + 1$ and therefore $d/r = m$ is an integer. □

We prove now the first part of Theorem 1.2 for chains. It serves as a basis of our reduction of chains.

Proposition 3.10. *Let $0 < d < r$. Then there is a natural bijection between $M^{ss}_{\text{ch}}(r, d)$ and $M^{ss}_{\text{ch}}(d, d - r)$. Analogous with stable chains.*

Proof. Let $\mathbf{a} = (a_1, \dots, a_r)$ be a semistable chain of type (r, d). It follows from Lemma 3.9 that its elements can take at most two consecutive values. Obviously, they can be only 0 or 1. From the inequality $a_1 \ge \mu(\mathbf{a}) > 0$ one gets $a_1 = 1$. Analogously $a_r = 1$. From the condition $\sum_{i=1}^r a_i - 1 = d$ we obtain that there are $d+1$ 1's among the elements of $\mathbf{a}$. Let $b_1, \dots, b_d$ be the lengths of consecutive zero-blocks between the 1's. We have $\sum_{i=1}^d b_i = r - d - 1$. Now, the chain $\mathbf{a}$ consisting of 0's and 1's is semistable if and only if the inequality from the definition 3.5 holds for any subchain starting and ending with a one. This can be written as follows. Any subchain $(b_{j+1}, \dots, b_{j+k})$ of the chain $(b_1, \dots, b_d)$ should satisfy

$$\frac{(k+1) - 1}{\sum_{i=j+1}^{j+k} b_i + k + 1} \le \frac{d}{\sum_{i=1}^d b_i + d + 1},$$

or, equivalently,

$$\frac{\sum_{i=j+1}^{j+k} b_i + 1}{k} \ge \frac{\sum_{i=1}^d b_i + 1}{d},$$

which can be written in the form

$$\frac{\sum_{i=j+1}^{j+k} (-b_i) - 1}{k} \le \frac{\sum_{i=1}^d (-b_i) - 1}{d}.$$

But this says precisely that the chain $(-b_1, -b_2, \dots, -b_d)$ is semistable. Its degree is $\sum_{i=1}^d (-b_i) - 1 = -(r - d - 1) - 1 = d - r$. The last thing to prove is that, conversely, any such semistable chain will give nonnegative numbers b_i so that we can reconstruct the chain $\mathbf{a}$. But the semistability condition for $(-b_1, \dots, -b_d)$ implies $-b_i - 1 \le (d - r)/d < 0$ and therefore $b_i \ge 0$. This altogether implies that there is a bijection between $M^{ss}_{\text{ch}}(r, d)$ and $M^{ss}_{\text{ch}}(d, d-r)$. The proof for stable chains goes through the same lines. □

This proposition shows that we can reduce the classification of (semi)stable chains of type (r, d) to the classification of (semi)stable chains of type $(d, d-r)$, i.e., of those with a smaller rank. The latter can be reduced to $M^{ss}_{\text{ch}}(d, r_0)$ (respectively, to $M^{s}_{\text{ch}}(d, r_0)$), where $0 \le r_0 < d$ by Lemma 3.6. Repeating these reductions we will finally end up with $M^{ss}_{\text{ch}}(h, 0)$ (respectively, $M^{s}_{\text{ch}}(h, 0)$), where $h = \gcd(r, d)$. So, the second part of Theorem 1.2 for chains should be proved (and classification should be done) only for the type $(h, 0)$.

For example, let us describe $M^{ss}_{\mathrm{ch}}(7,4)$. We write our reductions as follows

$$M^{ss}_{\mathrm{ch}}(7,4) \simeq M^{ss}_{\mathrm{ch}}(4,4-7) \simeq M^{ss}_{\mathrm{ch}}(4,1) \simeq M^{ss}_{\mathrm{ch}}(1,1-4) \simeq M^{ss}_{\mathrm{ch}}(1,0).$$

Thus, we take the unique element $(1) \in M^{ss}_{\mathrm{ch}}(1,0)$ and reconstruct the element from $M^{ss}_{\mathrm{ch}}(7,4)$ going from the right to the left in our sequence of isomorphisms. We get $(-2) \in M^{ss}_{\mathrm{ch}}(1,-3)$ and therefore the element of $M^{ss}_{\mathrm{ch}}(4,1)$ consists of two ones with a zero-block of length 2 between them, so we get $(1,0,0,1) \in M^{ss}_{\mathrm{ch}}(4,1)$. Then $(0,-1,-1,0) \in M^{ss}_{\mathrm{ch}}(4,-3)$ and the element of $M^{ss}_{\mathrm{ch}}(7,4)$ consists of five ones with zero-blocks of lengths $(0,1,1,0)$ between them, so we get $(1,1,0,1,0,1,1) \in M^{ss}_{\mathrm{ch}}(7,4)$.

Proposition 3.11. *The semistable chains of type $(r,0)$ are of the form*

$$(0,\dots,0,1,0,\dots,0,-1,0,\dots,0,1,\dots\dots,-1,0,\dots,0,1,0,\dots,0),$$

where 1 and -1 alternate and the zero-blocks are of arbitrary lengths (the whole sequence must be, of course, of length r). If $r>1$, none of these chains is stable. If $r=1$ there is precisely one semistable chain (1) and it is stable.

Proof. Let $\mathbf{a} = (a_1,\dots,a_r)$ be semistable of type $(r,0)$. Then we have $a_i - 1 \le \mu(\mathbf{a}) = 0$, $a_i + 1 \ge \mu(\mathbf{a}) = 0$ and therefore $-1 \le a_i \le 1$. Let there be k elements in $\mathbf{a}$ which are equal to 1 and l elements which are equal to -1. We have then $\deg \mathbf{a} = l - k - 1 = 0$ and so $l = k+1$. If there exists a subchain $\mathbf{b}$ containing only zeros and ones with at least two ones then $\deg \mathbf{b} \ge 2 - 1 > 0$ and therefore $\mu(\mathbf{b}) > \mu(\mathbf{a}) = 0$, which is impossible. This together with $l = k+1$ imply that 1 and -1 alternate in $\mathbf{a}$ and therefore $\mathbf{a}$ has a required form. Conversely, if a chain $\mathbf{a}$ has the form like in the condition of the proposition then, first of all, its degree equals 0. For any subchain $\mathbf{b}$ the difference between the numbers of 1's and -1's is not greater than 1 and therefore $\deg \mathbf{b} \le 0$, which implies $\mu(\mathbf{b}) \le \mu(\mathbf{a}) = 0$. To prove that $\mathbf{a}$ is not stable if $r > 1$ we notice that for a proper subchain (1) of $\mathbf{a}$ one has $\mu((1)) = 0 = \mu(\mathbf{a})$. The last assertion of the proposition is trivial. □

This proposition together with Proposition 3.10 implies Theorem 1.2 for chains. The further considerations are of independent interest.

Lemma 3.12. *A chain $\mathbf{a} = (a_1,\dots,a_r)$ is semistable (stable) if and only if for any of its extreme subchains $\mathbf{b}$ it holds $\mu(\mathbf{b}) \le \mu(\mathbf{a})$ ($\mu(\mathbf{b}) < \mu(\mathbf{a})$). In particular, if a chain $\mathbf{a}$ is non-stable then it contains an extreme destabilizing subchain.*

Proof. Assuming that for any extreme subchain $\mathbf{b}$ of $\mathbf{a}$ it holds $\mu(\mathbf{b}) \le \mu(\mathbf{a})$, we will show that $\mathbf{a}$ is semistable. Let $\mathbf{c} = (a_{k_1+1}, a_{k_1+2}, \dots a_{k_1+k_2})$ be a subchain of $\mathbf{a}$. We denote $k_3 = r - k_1 - k_2$, $x_1 = \sum_{i=1}^{k_1} a_i$, $x_2 = \sum_{i=k_1+1}^{k_1+k_2} a_i$, and $x_3 = \sum_{i=k_1+k_2+1}^{r} a_i$. We want to show that $\mu(\mathbf{c}) \le \mu(\mathbf{a})$, so we may suppose that $\mathbf{c}$ is not extreme, hence $k_1 \ge 1$ and $k_3 \ge 1$. The same proof as in Lemma 3.7 shows

$$\frac{x_1+x_2+x_3-1}{k_1+k_2+k_3} \le \frac{x_3}{k_3}, \quad \frac{x_1+x_2+x_3-1}{k_1+k_2+k_3} \le \frac{x_1}{k_1}$$

and therefore

$$\frac{x_1+x_2+x_3-1}{k_1+k_2+k_3} \le \frac{x_1+x_3}{k_1+k_3}.$$

This implies

$$\frac{x_2-1}{k_2} \le \frac{x_1+x_2+x_3-1}{k_1+k_2+k_3},$$

i.e., $\mu(\mathbf{c}) \le \mu(\mathbf{a})$. The proof for stable chains is analogous. □

Lemma 3.13. *Let* $\mathbf{a} = (a_1, \ldots, a_r)$ *be a semistable chain and* $\mathbf{b} = (a_1, \ldots, a_k)$ *be its extreme destabilizing subchain. Then the chains* $\mathbf{b}$ *and* $\mathbf{b}' = (a_{k+1}+1, a_{k+2}, \ldots, a_r)$ *are semistable chains with slope* $\mu(\mathbf{a})$.

Proof. It follows from the condition

$$\frac{\sum_{i=1}^{k} a_i - 1}{k} = \frac{(\sum_{i=1}^{k} a_i - 1) + \sum_{i=k+1}^{r} a_i}{r}$$

that

$$\mu(\mathbf{b}') = \frac{\sum_{i=k+1}^{r} a_i}{r-k} = \frac{(\sum_{i=1}^{k} a_i - 1) + \sum_{i=k+1}^{r} a_i}{r} = \mu(\mathbf{a}).$$

The semistability of $\mathbf{b}$ is trivial. To prove the semistability of $\mathbf{b}'$ we note that if $\mathbf{c}$ is a subchain of $\mathbf{b}'$ not containing the element $a_{k+1}+1$ then $\mu(\mathbf{c}) \le \mu(\mathbf{a}) = \mu(\mathbf{b}')$. If $\mathbf{c}$ contains $a_{k+1}+1$ then it is of the form $(a_{k+1}+1, a_{k+2}, \ldots, a_{k+l})$ and therefore it would follow from

$$\mu(\mathbf{c}) = \frac{\sum_{i=k+1}^{k+l} a_i}{l} > \mu(\mathbf{a}), \qquad \frac{\sum_{i=1}^{k} a_i - 1}{k} = \mu(\mathbf{a})$$

that

$$\frac{\sum_{i=1}^{k+l} a_i - 1}{k+l} > \mu(\mathbf{a}),$$

which is impossible as $\mathbf{a}$ is semistable. $\square$

We return to (semi)stable cycles.

Lemma 3.14. *The cycle* $\mathbf{a} = (a_1, a_2, \ldots a_r)$ *is semistable (stable) if and only if the cycle* $(a_1+1, a_2+1 \ldots, a_r+1)$ *is semistable (stable). In particular, there is a bijection* $M^{ss}_{\mathrm{cyc}}(r,d) \simeq M^{ss}_{\mathrm{cyc}}(r,d+r)$ *and we may always assume* $0 \le \deg a < r$.

Lemma 3.15. *If the chain* $\mathbf{a} = (a_1, \ldots, a_r)$ *is semistable then for any index* k *it holds* $\mu(\mathbf{a}) \le a_k + 1$.

Proof. Without loss of generality we may assume $k = r$. Semistability of $\mathbf{a}$ implies

$$(\sum_{i=1}^{r-1} a_i - 1)/(r-1) \le (\sum_{i=1}^{r} a_i)/r,$$

hence

$$\sum_{i=1}^{r} a_i \le r a_r + r$$

and the claim follows. $\square$

It follows that for any semistable cycle $\mathbf{a} = (a_1, \ldots, a_r)$ and any indices i, j it holds $a_i - 1 \le \mu(\mathbf{a}) \le a_j - 1$. As above, we obtain that the elements of $\mathbf{a}$ can take at most three consecutive values.

Lemma 3.16. *If a semistable cycle of type* (r,d) *contains elements with difference* 2 *then* d *is a multiple of* r *(hence* $d = 0$ *under the assumption* $0 \le d < r$*).*

Proof. The proof is the same as the proof of Lemma 3.9 $\square$

Proposition 3.17. *Let* $0 < d < r$. *Then there is a bijection between* $M^{ss}_{\mathrm{cyc}}(r,d)$ *and* $M^{ss}_{\mathrm{cyc}}(d, d-r)$. *Analogous with stable aperiodic cycles.*

Proof. Let $\mathbf{a} = (a_1, \dots, a_r)$ be a semistable cycle of type (r, d). We know that its elements can take at most two consecutive values. Obviously, they can be only 0 and 1. From the condition $\sum_{i=1}^{r} a_i = d$ we get that there are d ones among the elements of $\mathbf{a}$. Let $b_1, \dots, b_d$ be the lengths of consecutive zero-blocks between the ones. We have $\sum_{i=1}^{d} b_i = r - d$. Now, the cycle $\mathbf{a}$ consisting of zeros and ones is semistable if and only if the inequality from Definition 3.5 holds for any subchain starting and ending with a one. This can be written as follows. For any subchain $(b_{j+1}, \dots, b_{j+k})$ of the cycle $(b_1, \dots, b_d)$ one should have

$$\frac{(k+1)-1}{\sum_{i=j+1}^{j+k} b_i + k + 1} \le \frac{d}{\sum_{i=1}^{d} b_i + d},$$

or, equivalently,

$$\frac{\sum_{i=j+1}^{j+k} b_i + 1}{k} \ge \frac{\sum_{i=1}^{d} b_i}{d},$$

which can be written in the form

$$\frac{\sum_{i=j+1}^{j+k}(-b_i) - 1}{k} \le \frac{\sum_{i=1}^{d}(-b_i)}{d}.$$

But this says precisely that the cycle $(-b_1, -b_2, \dots, -b_d)$ is semistable. Its degree equals $\sum_{i=1}^{d}(-b_i) = d - r$. It remains to prove that, conversely, any such semistable cycle will produce nonnegative numbers b_i so that we can reconstruct the cycle $\mathbf{a}$. But the semistability condition implies $-b_i - 1 \le (d-r)/d < 0$, therefore $b_i \ge 0$. It is clear that the cycle $\mathbf{a}$ is aperiodic if and only if $\mathbf{b}$ is aperiodic. Altogether it implies that there is a bijection between $M^{ss}_{\mathrm{cyc}}(r, d)$ and $M^{ss}_{\mathrm{cyc}}(d, d-r)$. The proof for stable cycles goes through the same lines. $\square$

Using this proposition, precisely as it was done for chains, we can reduce the study of $M^{ss}_{\mathrm{cyc}}(r, d)$ to the study of $M^{ss}_{\mathrm{cyc}}(h, 0)$, where $h = \gcd(r, d)$.

For example, let us describe $M^{ss}_{\mathrm{cyc}}(7, 4)$. We write our reductions as follows

$$M^{ss}_{\mathrm{cyc}}(7,4) \simeq M^{ss}_{\mathrm{cyc}}(4, 4-7) \simeq M^{ss}_{\mathrm{cyc}}(4,1) \simeq M^{ss}_{\mathrm{cyc}}(1, 1-4) \simeq M^{ss}_{\mathrm{cyc}}(1,0).$$

Thus, we take the unique element $(0) \in M^{ss}_{\mathrm{cyc}}(1,0)$ and reconstruct the element from $M^{ss}_{\mathrm{cyc}}(7,4)$ going from the right to the left in our sequence of isomorphisms. We get $(-3) \in M^{ss}_{\mathrm{cyc}}(1,-3)$ and therefore the element of $M^{ss}_{\mathrm{cyc}}(4,1)$ equals $(1,0,0,0)$ (the length of zero-block equals 3). Then $(0,-1,-1,-1) \in M^{ss}_{\mathrm{cyc}}(4,-3)$ and the element of $M^{ss}_{\mathrm{cyc}}(7,4)$ has zero-blocks of lengths $(0,1,1,1)$, so it looks like $(1,1,0,1,0,1,0)$. Clearly, it is equivalent to $(1,0,1,0,1,0,1) \in M^{ss}_{\mathrm{cyc}}(7,4)$.

Proposition 3.18. *The semistable cycles of type* $(r, 0)$ *are of the form*

$$(0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0, 1, \dots\dots, -1, \dots, 0),$$

where 1 *and* -1 *alternate and zero-blocks are arbitrary (the sequence should of course be of length* r*). If* $r > 1$*, none of these cycles is stable aperiodic. If* $r = 1$ *there is just one semistable cycle* (1) *and it is stable.*

Proof. Let $\mathbf{a} = (a_1, \dots, a_r)$ be semistable of type $(r, 0)$. Then we have $a_i - 1 \le \mu(\mathbf{a}) = 0$, $a_i + 1 \ge \mu(\mathbf{a}) = 0$ and therefore $-1 \le a_i \le 1$. Let there be k elements in $\mathbf{a}$ which equal 1 and l elements which equal -1. We have then $\deg \mathbf{a} = l - k = 0$, so $l = k$. If there exists a subchain $\mathbf{b}$ containing only zeros and ones with at least two ones then $\deg \mathbf{b} \ge 2 - 1 > 0$ and therefore $\mu(\mathbf{b}) > \mu(\mathbf{a}) = 0$, which is impossible.

This, with $l = k$ imply that 1 and -1 alternate in $\mathbf{a}$ and therefore $\mathbf{a}$ has the required form. Conversely, if a chain $\mathbf{a}$ has the form like in the condition of the proposition then, first of all, its degree equals 0. For any subchain $\mathbf{b}$ the difference between the numbers of 1's and -1's is no greater than 1 and therefore $\deg \mathbf{b} \leq 0$, which implies $\mu(\mathbf{b}) \leq \mu(\mathbf{a}) = 0$. To prove that any aperiodic $\mathbf{a}$ is non-stable if $r > 1$ we notice that it contains nonzero elements, because otherwise it would be periodic. But for a proper subchain (1) of $\mathbf{a}$ one has $\mu((1)) = 0 = \mu(\mathbf{a})$, so $\mathbf{a}$ is non-stable. The last assertion of the proposition is trivial. $\square$

This proposition together with Proposition 3.17 implies Theorem 1.2 for cycles.

Lemma 3.19. *Let $\mathbf{a} = (a_1, \dots, a_r)$ be a semistable cycle and $\mathbf{b} = (a_1, \dots, a_k)$ be its destabilizing subchain. Then the chains $\mathbf{b}$ and $\mathbf{b}' = (a_{k+1}+1, a_{k+2}, \dots, a_{r-1}, a_r+1)$ are semistable chains with slope $\mu(\mathbf{a})$.*

Proof. It follows from the condition

$$\frac{\sum_{i=1}^k a_i - 1}{k} = \frac{(\sum_{i=1}^k a_i - 1) + (\sum_{i=k+1}^r a_i + 1)}{r}$$

that

$$\mu(\mathbf{b}') = \frac{\sum_{i=k+1}^r a_i + 1}{r-k} = \frac{(\sum_{i=1}^k a_i - 1) + (\sum_{i=k+1}^r a_i + 1)}{r} = \mu(\mathbf{a}).$$

The semistability of $\mathbf{b}$ is trivial. To prove the semistability of $\mathbf{b}'$ we note that if $\mathbf{c}$ is a subchain of $\mathbf{b}'$ not containing elements $a_{k+1}+1$ and a_r+1 then $\mu(\mathbf{c}) \leq \mu(\mathbf{a}) = \mu(\mathbf{b}')$. If $\mathbf{c}$ is a proper subchain of $\mathbf{b}'$ containing, say, $a_{k+1} + 1$ then it is of the form $(a_{k+1} + 1, a_{k+2}, \dots, a_{k+l})$ and therefore it would follow from

$$\mu(\mathbf{c}) = \frac{\sum_{i=k+1}^{k+l} a_i}{l} > \mu(\mathbf{a}), \qquad \frac{\sum_{i=1}^k a_i - 1}{k} = \mu(\mathbf{a})$$

that

$$\frac{\sum_{i=1}^{k+l} a_i - 1}{k+l} > \mu(\mathbf{a}),$$

which is impossible as $\mathbf{a}$ is semistable. $\square$

4. Classification of semistable sheaves

We know how to classify the (semi)stable chains and cycles, so the classification of (semi)stable sheaves will be complete if we will prove that it holds the converse of Corollaries 2.9 and 2.11. We do this in four steps.

Lemma 4.1. *The sheaf $\mathcal{B}(\mathbf{a})$ is stable if and only if the cycle $\mathbf{a}$ is stable. In this case degree and rank are coprime.*

Proof. The only if part is already proved. Let $\mathbf{a}$ be a stable cycle of type (r, d). We know that necessarily r and d are coprime and $\mathbf{a}$ is the unique stable cycle of type (r, d). There exist stable locally free sheaves of type (r, d) (see e.g. [1]). Let $\mathcal{B}(\mathbf{b}, m, \lambda)$ be any of them. Then $m = 1$ and $\mathbf{b}$ is stable of type (r, d), hence $\mathbf{b} = \mathbf{a}$. It follows that $\mathcal{B}(\mathbf{a})$ is stable. $\square$

Lemma 4.2. *The sheaf $\mathcal{S}(\mathbf{a})$ ($\mathbf{a} = (a_1, \dots, a_r)$) is stable if and only if the chain $\mathbf{a}' = (a_1+1, a_2, \dots, a_{r-1}, a_r+1)$ is stable. In this case degree and rank are coprime.*

Proof. The only if part is already proved. Let $\mathbf{a}'$ be a stable chain of type (r,d). Then r and d are coprime and $\mathbf{a}'$ is the unique stable chain of type (r,d). Let $M_C(r,d)$ denote the moduli space of stable sheaves of type (r,d) over C. The subspace of $M_C(r,d)$ consisting of the locally free sheaves $\mathcal{B}(\mathbf{b},1,\lambda)$ (where $\mathbf{b}$ is a unique stable cycle of type (r,d)) is isomorphic to k^*. It follows from the projectivity of $M_C(r,d)$ that it cannot coincide with k^* and therefore it contains some $\mathcal{S}(\mathbf{c})$, so that the corresponding chain $\mathbf{c}'$ of type (r,d) is stable and we deduce from the uniqueness of stable chains of type (r,d) that $\mathbf{c}'=\mathbf{a}'$ hence $\mathbf{c}=\mathbf{a}$ and $\mathcal{S}(\mathbf{a})$ is stable. □

Lemma 4.3. *The sheaf $\mathcal{S}(\mathbf{a})$ ($\mathbf{a}=(a_1,\dots,a_r)$) is semistable if and only if the chain $\mathbf{a}'=(a_1+1,a_2,\dots,a_{r-1},a_r+1)$ is semistable.*

Proof. The only if part is already proved. Conversely, if the chain $\mathbf{a}'$ is stable, then we are done. So, let us assume that $\mathbf{a}'$ is semistable but not stable. Then it contains an extreme destabilizing subchain, which without loss of generality we will assume to be of the form $(a_1+1,a_2,\dots,a_k)$. By Lemma 3.13, we know that the chains $(a_1+1,a_2,\dots,a_k)$ and $(a_{k+1}+1,a_{k+2},\dots,a_{r-1},a_r+1)$ are semistable with the same slope $\mu(\mathbf{a}')$, so by induction on rank we deduce that $\mathcal{S}((a_1,a_2,\dots,a_{k-1},a_k-1))$ and $\mathcal{S}((a_{k+1},a_{k+2},\dots,a_{r-1},a_r))$ are semistable with the slope $\mu(\mathbf{a}')$. Now, it follows from the exact sequence of Proposition 2.10 that $\mathcal{S}(\mathbf{a})$ is also semistable. □

Lemma 4.4. *The sheaf $\mathcal{B}(\mathbf{a})$ is semistable if and only if the cycle $\mathbf{a}$ is semistable.*

Proof. The only if part is already proved. Conversely, if the cycle $\mathbf{a}$ is stable, then we are done. So, let us assume that $\mathbf{a}$ is semistable but not stable. Then it contains a destabilizing subchain which, without loss of generality, we will assume to be of the form $(a_1,\dots,a_k)$. By Lemma 3.19, we know that the chains $(a_1,a_2,\dots,a_k)$ and $(a_{k+1}+1,a_{k+2},\dots,a_{r-1},a_r+1)$ are semistable with the same slope $\mu(\mathbf{a})$, therefore the sheaves $\mathcal{S}((a_1-1,a_2,\dots,a_{k-1},a_k-1))$ and $\mathcal{S}((a_{k+1},a_{k+2},\dots,a_{r-1},a_r))$ are semistable with the slope $\mu(\mathbf{a}')$. Now, it follows from the exact sequence of Proposition 2.8 that $\mathcal{B}(\mathbf{a})$ is also semistable. □

Altogether it proves Theorem 1.1. We formulate now some corollaries.

Corollary 4.5. *If $\gcd(r,d)>1$ then there are no stable sheaves in $\mathcal{E}(r,d)$. The number of non-locally free semistable sheaves in $\mathcal{E}(r,d)$ is finite and non-zero. The family of semistable locally free sheaves in $\mathcal{E}(r,d)$ is parameterized by a finite (non-empty) union of copies of k^*.*

Corollary 4.6. *If $\gcd(r,d)=1$ then all semistable sheaves in $\mathcal{E}(r,d)$ are stable. There is precisely one non-locally free semistable sheaf. The family of semistable locally free sheaves in $\mathcal{E}(r,d)$ is parameterized by k^**

Definition 4.7. We call a semistable sheaf F homogeneous if all the stable factors of its Jordan-Hölder filtration are isomorphic. The corresponding isomorphism class is called a basic block of F. In particular a stable sheaf is homogeneous.

Corollary 4.8. *All indecomposable semistable sheaves over C are homogeneous.*

Proof. First of all, we note that given two non-isomorphic stable sheaves G_1,G_2 of the same type (r,d), we have $\operatorname{Ext}^1(G_1,G_2)=0$. This follows immediately from

the Serre duality which is applicable because one of two sheaves G_1, G_2 is necessarily locally free (see Corollary 4.6). This implies, that also for any two homogeneous sheaves G_1, G_2 having non-isomorphic basic blocks of the same type one has $\mathrm{Ext}^1(G_1, G_2)$. Consider a Jordan-Hölder filtration of a given semistable sheaf F. If we have two consecutive non-isomorphic factors G_1, G_2 then we can change the filtration in such a way that G_1 and G_2 are interchanged (using $\mathrm{Ext}^1(G_1, G_2) = 0$). This shows that F has a filtration with homogeneous factors having pairwise different basic blocks. As we have shown, the Ext^1-group between two different factors is zero and therefore such a filtration necessarily splits. □

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Institut für Mathematik, Johannes Gutenberg-Universität Mainz, 55099 Mainz, Germany.

E-mail address: mozgov@mathematik.uni-mainz.de